Linear Programming (LP): Principles and Concepts

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Outline

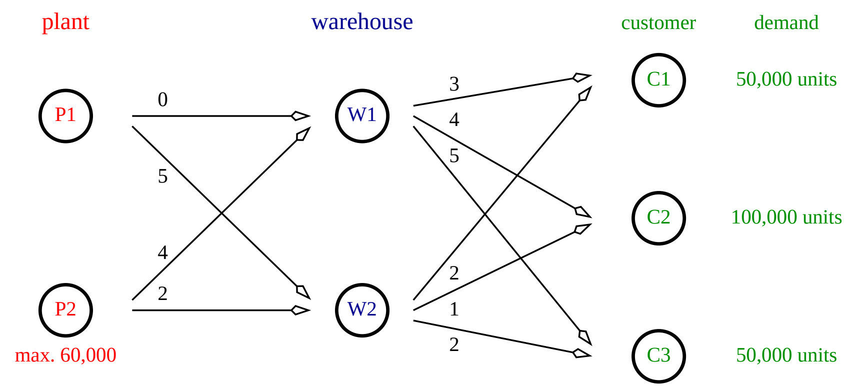
For additional details, see Rardin (1998), Chapter 5.1-5.2

Linear Programming

Multiple Choice Question: Why Study Linear Programming?

1. The prof cannot formulate nonlinear models
2. Linear programming is the most frequently used optimization method
3. Linear programming was developed in the 1940's
4. Formulation and solution knowledge will also help when addressing nonlinear models
5. Excellent software is available

Need for Optimization to Get the Best Solution



- The cost of manufacturing is the same
- The objective is thus to minimize the tranportation cost (using values in the figure)

What is the Best Plant?

Learning Goals

Attitudes

- An optimal solution is much better than an answer
- Numbers without understanding are useless

Skills

- Translate a complex problem into a mathematical formulation
- Solve an optimization problem!
- Communicate optimization results in "engineering terms"

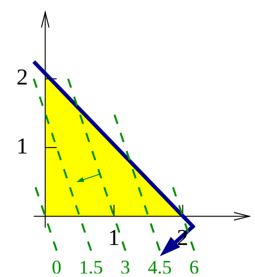
Knowledge

- When/how formulate LP models
- Analyze the LP solution sensitivity and diagnose "weird events"

Learning Goals

Challenge

Understand both the geometric interpretation (excellent understanding) and the matrix calculations (for the quantitative results) – At the same time!



$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,m} & a_{1,m+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,m} & a_{m,m+1} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

It Is Worth the Effort!

Linear Programs

Definition

An optimization model is a linear program (or LP) if it has:

1. continuous variables;
2. a single linear objective function; and
3. ony linear equality or inequality constraints

Standard notation for LP:

$$\begin{aligned} \min_{\mathbf{x}} \ & z \triangleq \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{s.t.} \ & \mathbf{A_h x} = \mathbf{b_h} \\ & \mathbf{A_g x} \leq \mathbf{b_g} \\ & \mathbf{x}_{\min} \leq \mathbf{x} \leq \mathbf{x}_{\max} \end{aligned}$$

- $x_j \triangleq j$th decision variable
- $c_j \triangleq$ objective function coefficient of x_j
- $a_{i,j} \triangleq$ constraint coefficient of x_j in the ith main constraint
- $b_i \triangleq$ right-hand side (RHS) constant term of main constraint i
- $m \triangleq$ number of main constraints
- $n \triangleq$ number of decision variables

Interior, Boundary and Extreme Points

Classification of the Points in an LP Feasible Region

A feasible solution to a linear program is said to be:

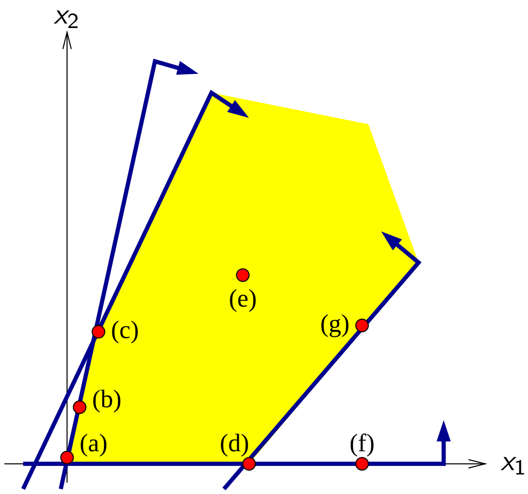
- a boundary point if at least one inequality constraint (that can be strict for some other feasible solutions) is satisfied as equality at the given point
- an interior point if no such inequalities are active
- an extreme point (or a corner point or a vertex) if every line segment in the feasible region containing it also has that point as an endpoint

Important Remarks About Extreme Points:

- Extreme point are special boundary points so-named because they "stick out" (due to convexity of the feasible region of an LP)
- Every extreme point is determined by a set of constraints that are active *only* at that solution

Classifying Feasible Points

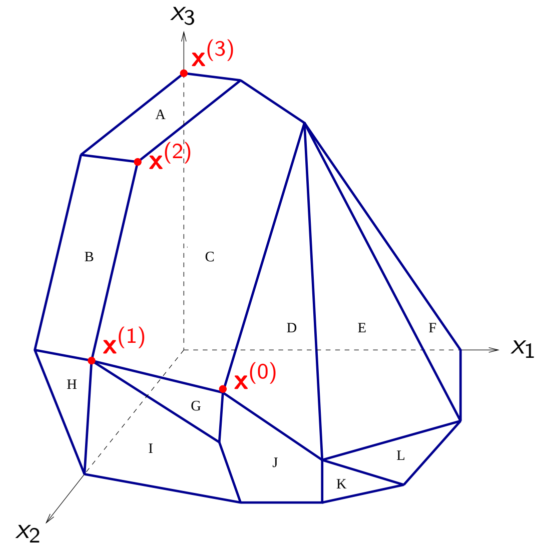
Class Exercise: Classify the labeled solutions as interior, boundary, and/or extreme points of the following LP feasible region:



(a)
(b)
(c)
(d)
(e)
(f)
(g)

Extreme Points

Class Exercise: List all sets of 3 constraints determining points $x^{(0)}$, $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$



$x^{(0)}$:

$x^{(1)}$:

$x^{(2)}$:

$x^{(3)}$:

Adjacent Extreme Points and Edges

Adjacent Extreme Points

Two extreme points of an LP feasible region are adjacent if they are determined by active constraint sets differing in *only one* element.

Edge

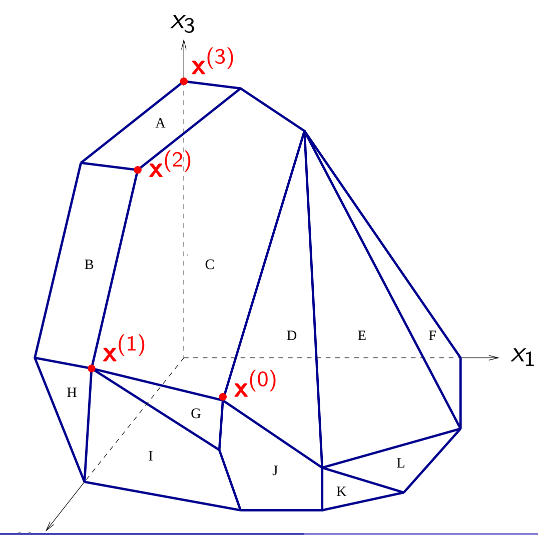
An edge of an LP feasible region is a 1-dimensional set of feasible points along a line determined by a collection of active constraints.

Remark:

- Adjacent extreme points are joined by an edge determined by the active constraints the extreme points have in common

Adjacent Extreme Points

Class Exercise: Which points among $x^{(0)}$, $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$ are adjacent extreme points? Which active constraints do these points have in common?

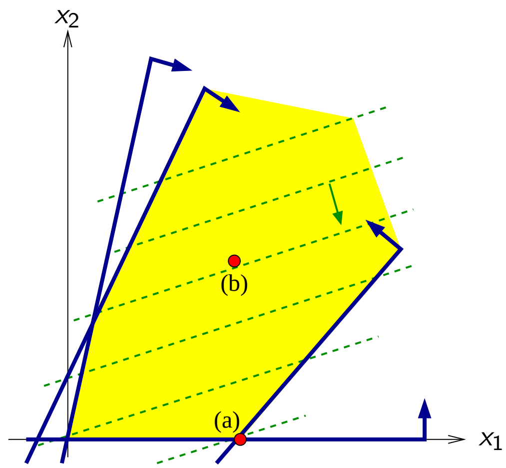


- Adjacent Extreme Points:
- Common Active Constraints:

Optimal Points in LP

Fundamental Result

Every optimal solution to an LP (that has a non-constant objective function) will be a boundary point of its feasible region



Class Exercise: Explain why point (b) cannot be optimal? How about point (a)?

Optimal Points in LP

Fundamental Result

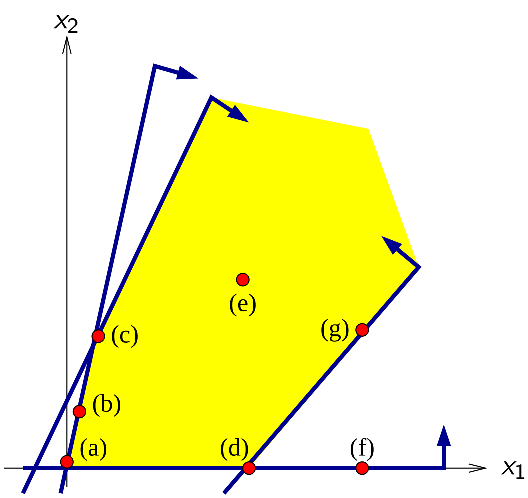
Every optimal solution to an LP (that has a non-constant objective function) will be a boundary point of its feasible region

Other Fundamental Properties:

1. If an LP has *any* extreme point, then one such point must be an optimal solution
   Why?
2. If an LP has a *unique* optimal solution, that optimum must occur at an extreme point of its feasible region
   Why?
3. *Every* local optimum for an LP is a global optimum
   Why?

Identifying Optimal Points

Class Exercise: Indicate which of the labeled points can be optimal or uniquely optimal for some objective function:



(a)
(b)
(c)
(d)
(e)
(f)
(g)

Naive Algorithm

Exploiting the extreme point concept as the foundation for a solution method:

Algorithm:

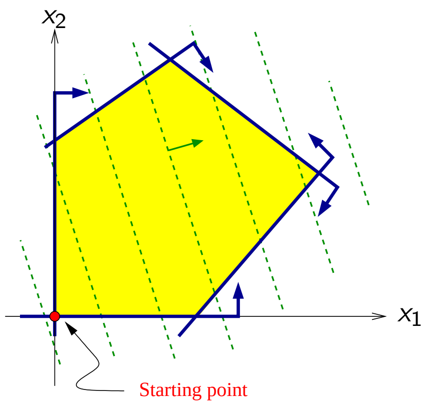
Calculate the objective function value at *all* extreme points, and keep whichever is best!

Application: Consider an LP comprising 10 variables and 20 inequality constraints.

Up to how many extreme points exist for this (rather small) LP problem?

Back to the Drawing Board!

Class Exercise: Develop an algorithm that combines the basic numerical approach with the key geometric insight



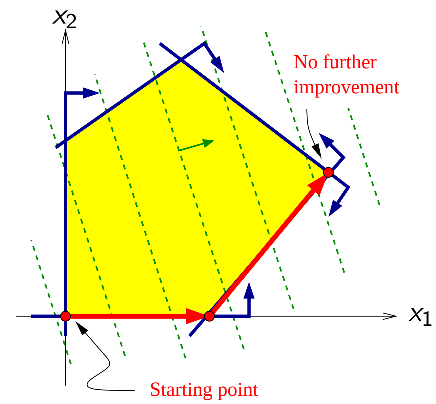
Basic Numerical Optimization:

- Use local information *only*, in an iterative scheme
- Determine a feasible next point that improves the objective value
- check if further improvement is possible: if 'yes', continue; else, stop

Basis of the Simplex Algorithm!

Back to the Drawing Board!

Class Exercise: Develop an algorithm that combines the basic numerical approach with the key geometric insight



One possible approach:

- Consider *only* adjacent extreme points for improvement direction
- Move along the edge that yields the greatest rate of improvement
- Move until another extreme point has been reached
- Check if further improvement is possible: if 'yes', continue; else, stop

Basis of the Simplex Algorithm!

LP Standard Form

Definition:

LP in standard form have:

- only *equality* main constraints
- only *nonnegative* variables
- objective function and main constraints simplified so that each variable appears at most once (on the left-hand side), and any constant term (possibly zero) appears on the right-hand side

$$\min_{\mathbf{x}} \; z \triangleq \mathbf{c}^{\mathsf{T}}\mathbf{x}$$
$$\text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

$x_j \triangleq$ jth decision variable

$c_j \triangleq$ objective function coefficient of x_j

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$n \triangleq$ number of decision variables

LP Standard From: Why and How?

Why LP in Standard Form?

-
-

How to Yield LP Standard Form?

1. Convert $\leq$ and $\geq$ inequalities to equalities
2. Convert nonpositive variables to nonnegative
3. Convert unrestricted (URS) variables to nonnegative

LP Standard From: Some Practice!

Class Exercise:

1. Place each of the following LP in standard form
2. Identify the m, n, $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$ of standard matrix representation

$$
\begin{aligned}
\min \quad & 9w_1 + 6w_2 \\
\text{s.t.} \quad & 2w_1 + w_2 \geq 10 \\
& w_1 \leq 50 \\
& w_1 + w_2 = 40 \\
& 100 \geq w_1 + 2w_2 \geq 15 \\
& w_1 \geq 0, w_2 \leq 0
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & 15(2x_1 + 8x_2) - 4x_3 \\
\text{s.t.} \quad & 2(10 - x_1) + x_2 + 5(9 - x_3) \geq 10 \\
& x_1 + 2x_2 \leq x_3 \\
& x_1, x_2 \geq 0, x_3 \text{ URS}
\end{aligned}
$$