Linear Programming (LP): Sensitivity Analysis

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ChE 4G03: Optimization in Chemical Engineering

Linear Programming

Problem Statement

$$\begin{aligned} \max \quad & z \triangleq x_1 + x_2 \\ \text{s.t.} \quad & -x_1 + x_2 \geq 0 \\ & x_1 \leq 2 \\ & x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Simplex $\Longrightarrow$ Search

Solution

$$\begin{aligned} z &= 5 && \rightarrow \text{Optimal value} \\ -x_1 + x_2 &= 1 \geq 0 && \rightarrow \text{Inactive} \\ x_1 &= 2 \leq 2 && \rightarrow \text{Active} \\ x_2 &= 3 \leq 3 && \rightarrow \text{Active} \\ x_1 &= 2 \geq 0 && \rightarrow \text{Inactive} \\ x_2 &= 3 \geq 0 && \rightarrow \text{Inactive} \end{aligned}$$

Is there more about the solution that we would like to know?

- Mathematical optima do not suffice because such data as costs, profits, yields, supplies and demands are always uncertain (at the time the model is solved) — Sometimes within a factor of 10!
- How can we trust the outcome of imperfectly parameterized models?
  - Maybe a cost and demand controls everything, or has no impact?

Linear Programming

Yes, **sensitivity** or **"what if"** analysis!

- Mathematical optima simply provide a best choice of decision variables for *one* fixing of the inputs (nominal values)
- Sensitivity analysis then tries to complete the picture by studying how results would vary with changes in parameter values

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Example of Sensitivity Information

What is the effect of a change in this parameter on the optimal cost z^* and on the optimal solution point $\mathbf{x}^* = (x_1^* \ x_2^*)$?

Linear Programming

Yes, **sensitivity** or **"what if"** analysis!

- Mathematical optima simply provide a best choice of decision variables for *one* fixing of the inputs (nominal values)
- Sensitivity analysis then tries to complete the picture by studying how results would vary with changes in parameter values
  - We could always change parameter values and resolve the LP — This is a lot of work for large models!
  - Sensitivity analysis yields useful information with little computation and helps build insight

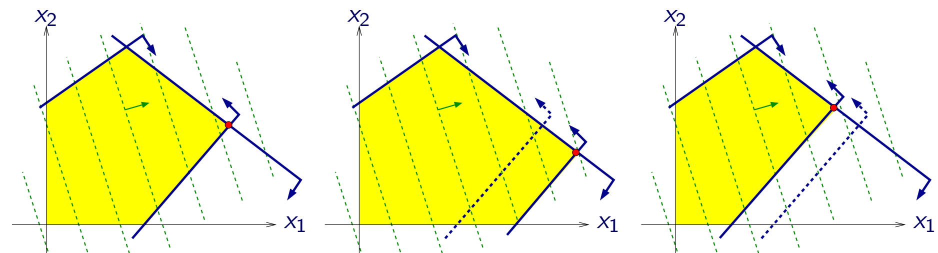
		Parameter Change	Structure Change
Qualitative Sensitivity	– direction of z change	**Yes**	**Yes**
Quantitative Sensitivity	– magnitude of change in z – limited range of validity	**Yes**	**No**

Outline

For additional examples, see Rardin (1998), Chapter 7

Qualitative Sensitivity: Relaxing vs. Tightening Constraints

1. **Relaxing** the constraints of an optimization model either makes the optimal value better (higher for a maximize, lower for a minimize) or leave it unchanged.
2. **Tightening** the constraints of an optimization model either makes the optimal value worse or leave it unchanged.



(a) Nominal (b) Relaxed (c) Tightened

Warning: Tightening the constraints too much can make the problem become infeasible!

Qualitative Sensitivity: Relaxing vs. Tightening Constraints

Class Exercise: For each of the following *single* changes to a *maximization* problem, indicate the effect on the optimal solution value:

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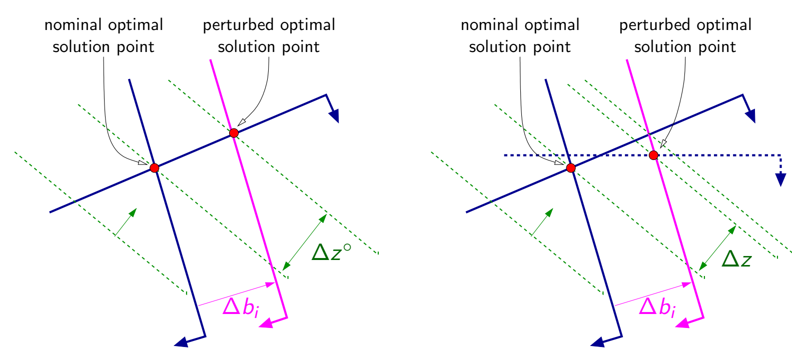
- Change the RHS of the 1st constraint to -1
- Eliminate the 2nd constraint
- Add the constraint $7x_1 + 3x_2 \leq 8$
- Allow variable x_2 to be unrestricted (URS)
- Change the 1st constraint to an equality
- Change the coefficient of x_1 in the LHS of the 1st constraint to -2

Change in Feasible Region: Direction of Change

1. The optimal solution value z^* changes monotonically as the feasible region is enlarged (constraint relaxation) or reduced (tightening)
2. The rate of change of z^* decreases *in absolute value* as the feasible region increases (no matter if active constraints change or not)

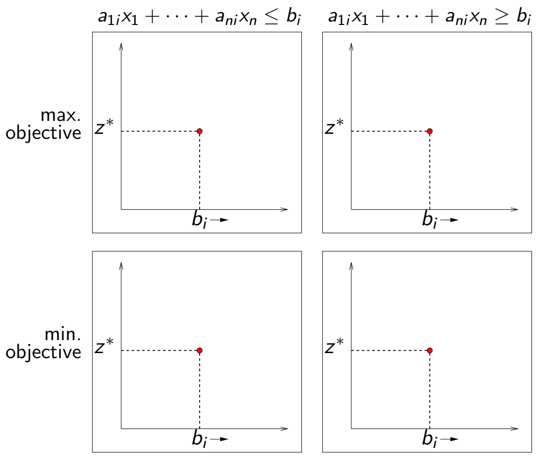


(a) without change in active set $\frac{\Delta z^\circ}{\Delta b_i} \triangleq \nu_i^\circ$

(b) with change in active set $\frac{\Delta z}{\Delta b_i} \triangleq \nu_i < \nu_i^\circ$

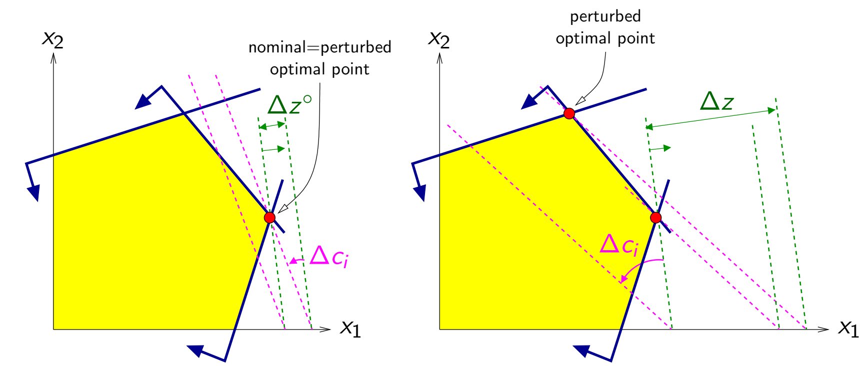
Change in Feasible Region: Direction of Change

Class Exercise: Sketch the typical qualitative behavior of the optimal solution value z^* as the RHS value b_i of an active constraint is increased or decreased from the nominal case; assume that the active set changes over the range of b_i — See Rardin (1998) for further discussion



Change in Cost Coefficients: Direction of Change

1. Over a limited range of cost coefficients c_i, the actual extreme point $\mathbf{x}^*$ (optimal solution) does not change
2. Changes that help the optimal value z^* help more and more as the change becomes bigger (no matter if active constraints change or not)



(a) without change in active set $\frac{\Delta z^\circ}{\Delta c_i} \triangleq \xi_i^\circ$

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Change in Cost Coefficients: Direction of Change

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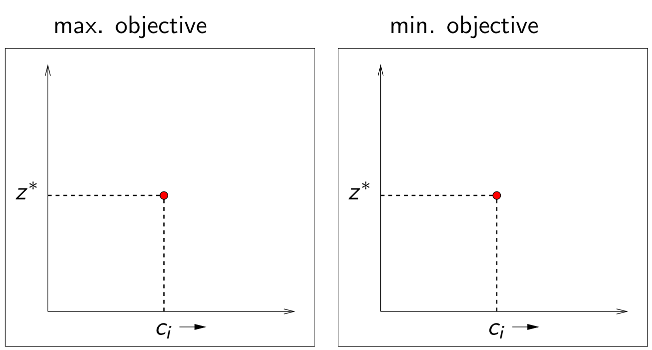


Quantitative Sensitivity

1. Quantitative sensitivity analysis is limited to optima with the same active constraints as the nominal case.
2. Quantitative sensitivity analysis provides an exact value of the sensitivity $\frac{\Delta z^*}{\Delta \text{parameter}}$.
3. Quantitative sensitivity analysis provides the range of parameter changes which result in no change to the active constraint.

Marginal Price

The change in optimal solution value z^* per unit increase in the right-hand side b_i of a main constraint $\sum_{j=1}^n a_{ij}x_j \leq, =, \geq b_i$ is called a marginal price (or shadow price or dual variable)

- There is one marginal price ν_i for each main constraint

Constraint i is:	$\leq$	$\geq$	$=$
min. objective	$\nu_i \leq 0$	$\nu_i \geq 0$	ν_i URS
max. objective	$\nu_i \geq 0$	$\nu_i \leq 0$	ν_i URS

Calculating Marginal Prices

- Consider the partitioning of $\mathbf{x}$, $\mathbf{A}$, $\mathbf{c}$ into basic and nonbasic components (at the nominal solution):

$$\mathbf{x}^* \triangleq [\mathbf{x}^*_{\rm B} \mid \mathbf{x}^*_{\rm NB}], \qquad \mathbf{A} \triangleq [\mathbf{A}^*_{\rm B} \mid \mathbf{A}^*_{\rm NB}], \qquad \mathbf{c} \triangleq [\mathbf{c}^*_{\rm B} \mid \mathbf{c}^*_{\rm NB}]$$

- If the active constraints *remain identical*, a change Δb_i in the RHS of constraint i induces basic/nonbasic variable changes as:

$$\Delta \mathbf{x}_{\rm NB} = \mathbf{0}$$
$$\Delta \mathbf{x}_{\rm B} = (\mathbf{A}^*_{\rm B})^{-1} \begin{pmatrix} 0 & \cdots & 0 & \Delta b_i & 0 & \cdots & 0 \end{pmatrix}^{\rm T}$$

Why?

- The marginal price ν_i is therefore obtained as:

$$\nu_i = \mathbf{c}^{\rm T}_{\rm B} (\mathbf{A}^*_{\rm B})^{-1} \begin{pmatrix} 0 & \cdots & 0 & \Delta b_i & 0 & \cdots & 0 \end{pmatrix}^{\rm T}$$

Why?

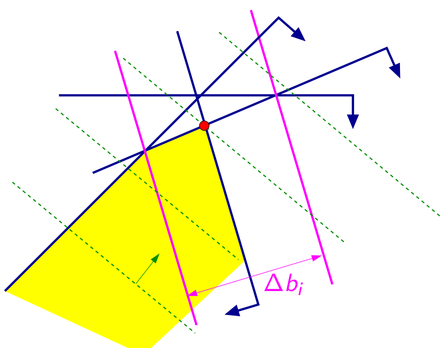
Calculating Marginal Prices

Class Exercise: For the following LP problem, what are the marginal prices for unit changes in the RHS of the first and second constraints?

	x_1	x_2	x_3	x_4	x_5	
max $\mathbf{c}$	1	1	0	0	0	$\mathbf{b}$
$\mathbf{A}$	-1	1	-1	0	0	0
	1	0	0	1	0	2
	0	1	0	0	1	3
$\mathscr{B}^*$	B	B	B	N	N	
$\mathbf{x}^*$	2	3	1	0	0	$z^* = 5$

Range of Validity of Marginal Prices

- How much can the right-hand side b_i of constraint i be changed without modifying the set of active constraints?



- Both the optimal point $\mathbf{x}^*$ and optimal solution value $\mathbf{z}^*$ change, even though the active set does not change
- On the other hand, the marginal price ν_i does not change when b_i remains in the allowable range

Sensitivity to Change in Cost Coefficients

In the case that the active constraints **remain identical**:

- A change Δc_i in the cost coefficient of variable x_i incurs a change in optimal cost as:

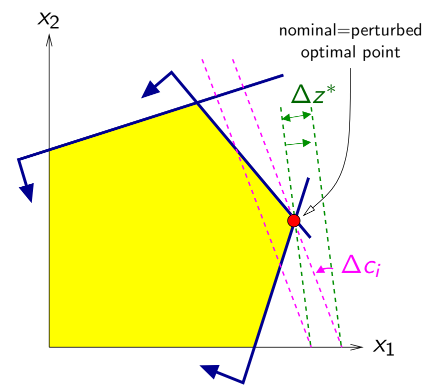
$\Delta z^* = 0$ if x_i is nonbasic — **Why?**

$\Delta z^* = x_i^* \Delta c_i$ if x_i is basic

- The solution point remains identical:

$$\Delta \mathbf{x}^* = \mathbf{0}$$

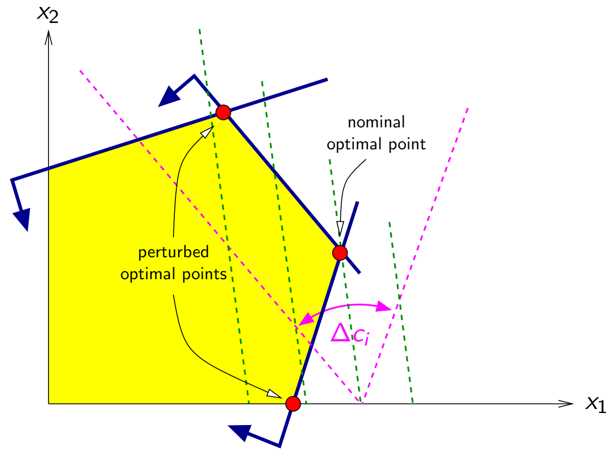
- The feasible region remains unaffected



No change in active set

Range of Validity of Cost Sensitivity

- How much can the cost coefficient c_i of variable i be changed without modifying the set of active constraints?



Computer Outputs and "What If" Questions

Software products (such as the solvers in GAMS) report:

1. the marginal price and the associated range of validity for *every* constraint
2. the sensitivity to change in cost coefficient and the associated range of validity

Outside of this range, the set of active constraints changes — **What can we do?**

Typical (Constraint) Sensitivity Analysis Output:

Constraint ID	Status	Slack	Marginal Price	Lower Range	Upper Range
Max. reflux flow	Active	0	3.74	123	47
Max. Pump 7	Inactive	321	0	321	1.0E+30

Answering "What If" Questions (for Single Parameter)

Class Exercise: Use the table on the preceeding slide to answer the following questions:

- **What is the effect on the profit of changing the reflux flow rate capacity by 2.1?**
- **Are the slack and marginal price values consistent for Pump 7?**
- **What does a maximum allowable increase of $1.0E + 30$ mean?**

Answering "What If" Questions (for Single Parameter)

Class Exercise: Use the table on the preceeding slide to answer the following questions:

- **A proposal has been made to increase the capacity of Pump 7. What is your response to the proposal?**
- **What is the effect of reducing the reflux flow by 125?**

The Case of Multiple Parameter Changes

In certain cases, it is possible to predict the effect of changing:

- multiple RHS coefficients b_i; or
- multiple cost coefficients c_j

The 100% Rule (Worst-Case Rule)

Consider the percentage of allowable change being made, and add all these percentages.

- If the sum is no more than 100%, then the current active constraints will remain identical
- If the sum is more than 100%, the current active constraints may not remain optimal

Caution:

The 100% rule does not apply to mixed changes in the constraint RHS coefficients b_i and the cost coefficients c_j

Answering "What If" Questions (for Multiple Parameters)

Class Exercise: Answer the following questions based on the data given in the table:

Constraint ID	Status	Slack	Marginal Price	Lower Range	Upper Range
Max. reflux flow	Active	0	3.74	123	47
Max. Pump 7	Inactive	321	0	321	1.0E+30
Max. Pump 5	Active	0	1.27	55	23

- **What is the effect on the profit of changing the reflux flow rate capacity by 35 and reducing the capacity of Pump 5 by 14 at the same time?**

Quantitative Sensitivity Analysis in a Nutshell

Question that can sometimes be answered with quantitative sensitivity analysis:

- Change in a single RHS coefficient b_i or cost coefficient c_j
- Change in multiple RHS coefficients
- Change in multiple cost coefficients

If quantitative sensitivity analysis does not apply, the model has to be resolve for quantitative results:

- Single large changes in RHS or cost coefficients
- Multiple changes in RHS and cost coefficients simultaneously
- Changes in LHS coefficients (**A** matrix)

$$\begin{aligned} \max \quad & z \triangleq \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{s.t.} \quad & \boxed{\mathbf{A}}\,\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Sensitivity Analysis — The Final Words

- We have covered the interpretation of the qualitative and quantitative sensitivity results for LP.

These are critical skills for the engineer!

- We have not covered the details regarding the computation of quantitative sensitivity results.

For an introduction to the dual formulation, see Rardin (1998)